

Cubic Maximal Nontraceable Graphs*

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Abstract

We determine a lower bound for the number of edges of a 2-connected maximal nontraceable graph, and present a construction of an infinite family of maximal nontraceable graphs that realize this bound.

Keywords: maximal nontraceable, maximal nonhamiltonian, hypohamiltonian, hamiltonian path, hamiltonian cycle, traceable, nontraceable, hamiltonian, nonhamiltonian, maximal hypohamiltonian

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1 Introduction

We consider only simple, finite graphs G and denote the vertex set and edge set of G by $V(G)$ and $E(G)$, respectively. The *open neighbourhood* of a vertex v in G is the set $N_G(v) = \{x \in V(G) : vx \in E(G)\}$. If U is a nonempty subset of $V(G)$ then $\langle U \rangle$ denotes the subgraph of G induced by U .

A graph G is *hamiltonian* if it has a *hamiltonian cycle* (a cycle containing all the vertices of G), and *traceable* if it has a *hamiltonian path* (a path containing all the vertices of G). A graph G is *maximal nonhamiltonian* (MNH) if G is not hamiltonian, but $G + e$ is hamiltonian for each $e \in E(\overline{G})$, where \overline{G} denotes the complement of G . A graph G is *maximal nontraceable* (MNT) if G is not traceable, but $G + e$ is traceable for each $e \in E(\overline{G})$. A graph G is *hypohamiltonian* if G is not hamiltonian, but every vertex-deleted subgraph $G - v$ of G is hamiltonian. We say that a graph G is *maximal hypohamiltonian* (MHH) if it is MNH and hypohamiltonian.

In 1978 Bollobás [1] posed the problem of finding the least number of edges, $f(n)$, in a MNH graph of order n . Bondy [2] had already shown that a MNH graph with order $n \geq 7$ that contained m vertices of degree 2 had at least $(3n + m)/2$ edges, and hence $f(n) \geq \lceil 3n/2 \rceil$ for $n \geq 7$. Combined results of

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Clark, Entringer and Shapiro [4], [5] and Lin, Jiang, Zhang and Yang [9] show that $f(n) = \lceil 3n/2 \rceil$ for $n \geq 19$ and for $n = 6, 10, 11, 12, 13, 17$. The values of $f(n)$ for the remaining values of n are also given in [9].

Let $g(n)$ be the minimum size of a MNT graph of order n . Dudek, Katona and Wojda [7] showed that $g(n) \geq (3n - 20)/2$ for all n and, by means of a recursive construction, they found MNT graphs of order n and size $O(n \log n)$. To date, no cubic MNT graphs have been reported. We construct an infinite family of cubic MNT graphs, thus showing that $g(n) \leq 3n/2$ for infinitely many n .

Now let $g_2(n)$ be the minimum size of a 2-connected MNT graph of order n . We prove that $g_2(n) \geq \lceil 3n/2 \rceil$ for $n \geq 7$. It then follows from our constructions that $g_2(n) = \lceil 3n/2 \rceil$ for $n = 8p$ for $p \geq 5$, $n = 8p + 2$ for $p \geq 6$, $n = 8p + 4$ for $p = 3$ and $p \geq 6$, and $n = 8p + 6$ for $p \geq 4$.

2 A lower bound for the size of a 2-connected MNT graph

Bondy [2] proved that if G is a 2-connected MNH graph and $v \in V(G)$ with degree $d(v) = 2$, then each neighbour of v has degree at least 4. He also showed that the neighbours of such a vertex are in fact adjacent.

In order to prove a corresponding result for 2-connected MNT graphs we need the following result.

Lemma 2.1 *Let Q be a path in a MNT graph G . If $\langle V(Q) \rangle$ is not complete, then some internal vertex of Q has a neighbour in $G - V(Q)$.*

Proof. Let u and v be two nonadjacent vertices of $\langle V(Q) \rangle$. Then $G + uv$ has a hamiltonian path P . Let x and y be the two endvertices of Q and suppose no internal vertex of Q has a neighbour in $G - V(Q)$. Then P has a subpath R in $\langle V(Q) \rangle + uv$ and R has either one or both endvertices in $\{x, y\}$. If R has only one endvertex in $\{x, y\}$, then P has an endvertex in Q . In either case the path obtained from P by replacing R with Q is a hamiltonian path of G . ■

Lemma 2.2 *If G is a MNT graph and $v \in V(G)$ with $d(v) = 2$, then the neighbours of v are adjacent. If in addition G is 2-connected, then each neighbour of v has degree at least 4.*

Proof. Let $N_G(v) = \{x_1, x_2\}$ and let Q be the path x_1vx_2 . Since $N_G(v) \subseteq Q$, it follows from Lemma 2.1 that $\langle V(Q) \rangle$ is a complete graph; hence x_1 and x_2 are adjacent.

Now assume that G is 2-connected. Since G is not traceable we assume $d(x_1) > 2$. Then also $d(x_2) > 2$ otherwise x_1 would be a cut vertex of G .

Let z be a neighbour of x_1 and let Q be the path zx_1vx_2 . Since $d(v) = 2$ the graph $\langle V(Q) \rangle$ is not complete, and hence it follows from Lemma 2.1 that x_1 has a neighbour in $G - V(Q)$. Thus $d(x_1) \geq 4$. Similarly $d(x_2) \geq 4$. ■

We also have the following two lemmas concerning MNT graphs that have vertices of degree 2.

Lemma 2.3 *Suppose G is a 2-connected MNT graph. Suppose $v_1, v_2 \in V(G)$ such that $d(v_1) = d(v_2) = 2$ and v_1 and v_2 have exactly one common neighbour x . Then $d(x) \geq 5$.*

Proof. The vertices v_1 and v_2 cannot be adjacent otherwise x would be a cut vertex. Let $N(v_i) = \{x, y_i\}$; $i = 1, 2$. It follows from Lemma 2.2 that x is adjacent to y_i ; $i = 1, 2$. Let Q be the path $y_1v_1xv_2y_2$. Since $\langle V(Q) \rangle$ is not complete, it follows from Lemma 2.1 that x has a neighbour in $G - V(Q)$. Hence $d(x) \geq 5$. ■

Lemma 2.4 *Suppose G is a MNT graph. Suppose $v_1, v_2 \in V(G)$ such that $d(v_1) = d(v_2) = 2$ and v_1 and v_2 have the same two neighbours x_1 and x_2 . Then $N_G(x_1) - \{x_2\} = N_G(x_2) - \{x_1\}$. Also $d(x_1) = d(x_2) \geq 5$.*

Proof. From Lemma 2.2 it follows that x_1 and x_2 are adjacent. Let Q be the path $x_2v_1x_1v_2$. $\langle V(Q) \rangle$ is not complete since v_1 and v_2 are not adjacent. Thus it follows from Lemma 2.1 that x_1 has a neighbour in $G - V(Q)$. Now suppose $p \in N_{G-V(Q)}(x_1)$ and $p \notin N_G(x_2)$. Then a hamiltonian path P in $G + px_2$ contains a subpath of either of the forms given in the first column of Table 1. Note that $i, j \in \{1, 2\}$; $i \neq j$ and that L represents a subpath of P in $G - \{x_1, x_2, v_1, v_2, p\}$. If each of the subpaths is replaced by the corresponding subpath in the second column of the table we obtain a hamiltonian path P' in G , which leads to a contradiction.

Subpath of P	Replace with
$v_ix_1v_jx_2p$	$v_ix_2v_jx_1p$
$v_ix_1Lpx_2v_j$	$v_ix_2v_jx_1Lp$

Table 1

Hence $p \in N_G(x_2)$. Thus $N_G(x_1) - \{x_2\} \subseteq N_G(x_2) - \{x_1\}$. Similarly $N_G(x_2) - \{x_1\} \subseteq N_G(x_1) - \{x_2\}$. Thus $N_G(x_1) - \{x_2\} = N_G(x_2) - \{x_1\}$ and hence $d(x_1) = d(x_2)$. Now let Q be the path $px_1v_1x_2v_2$. Since $\langle V(Q) \rangle$ is not complete, it follows from Lemma 2.1 that x_1 or x_2 has a neighbour in $G - V(Q)$. Hence $d(x_1) = d(x_2) \geq 5$. ■

We now consider the size of a 2-connected MNT graph.

Lemma 2.5 *Suppose G is a MNT graph of order $n \geq 6$ and that v_1, v_2 and v_3 are vertices of degree 2 in G having the same neighbours, x_1 and x_2 . Then $G - \{v_1, v_2, v_3\}$ is complete and hence $|E(G)| = \frac{1}{2}(n^2 - 7n + 24)$.*

Proof. Suppose $G - \{v_1, v_2, v_3\}$ is not complete. Then there exist $p, q \in V(G) - \{v_1, v_2, v_3\}$ which are not adjacent. However, since $\{v_1, v_2, v_3\}$ is an independent set, no path in $G + pq$ having pq as an edge can contain all three of v_1, v_2 and v_3 . Thus $G + pq$ is not traceable. Thus $G - \{v_1, v_2, v_3\} = K_{n-3}$. Hence $|E(G)| = \frac{1}{2}(n-3)(n-4) + 6$. ■

Theorem 2.6 *Suppose G is a 2-connected MNT graph. If G has order $n \geq 7$ and m vertices of degree 2, then $|E(G)| \geq \frac{1}{2}(3n + m)$.*

Proof. If G has three vertices of degree 2 having the same two neighbours then, by Lemma 2.5

$$|E(G)| = \frac{1}{2}(n^2 - 7n + 24) \geq \frac{1}{2}(3n + m) \text{ when } n \geq 7,$$

since $m = 3$.

We now assume that G does not have three vertices of degree 2 that have the same two neighbours. Let v_1, \dots, v_m be the vertices of degree 2 in G and let $H = G - \{v_1, \dots, v_m\}$. Then by Lemmas 2.2, 2.3 and 2.4 the minimum degree, $\delta(H)$ of H is at least 3. Hence

$$|E(G)| = |E(H)| + 2m \geq \frac{3}{2}(n - m) + 2m = \frac{1}{2}(3n + m).$$

■

Thus $g_2(n) \geq \frac{1}{2}(3n + m)$ for $n \geq 7$. For $m \geq 1$ this bound is realized for $n = 7$ (a Zelinka Type I graph [10]) and $n = 18$ (a graph constructed in [3]). These graphs are depicted in Fig. 1.

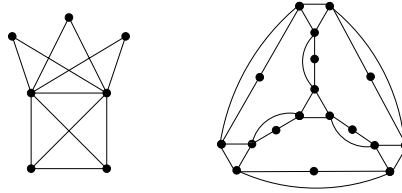


Fig. 1

We now construct an infinite family of 2-connected cubic MNT graphs, showing that $g_2(n) = \frac{3n}{2}$ for infinitely many n .

3 A construction of an infinite family of cubic MNT graphs

In order to construct cubic MNT graphs we require the following lemmas concerning MHH graphs.

Lemma 3.1 Suppose H is a hypohamiltonian graph having a vertex z of degree 3. Put $F = H - z$.

- (a) F has a hamiltonian path ending at any of its vertices.
- (b) There is no hamiltonian path in F with both endvertices in $N_H(z)$.
- (c) For any $y \in N_H(z)$ there exists a hamiltonian path in $F - y$ with the other two vertices of $N_H(z)$ being the endvertices.

Proof.

- (a) F is hamiltonian.
- (b) If a hamiltonian path exists in F having both endvertices in $N_H(z)$, then H has a hamiltonian cycle, which is a contradiction.
- (c) Since $H - y$ is hamiltonian there is a hamiltonian cycle in $H - y$ containing the path vzw , where $v, w \in N_H(z) - \{y\}$. Thus there is a hamiltonian path in $F - y$ with endvertices v and w .

■

Lemma 3.2 Suppose H is a MNH graph having a vertex z of degree 3. Put $F = H - z$. If u_1 and u_2 are nonadjacent vertices in F , then $F + u_1u_2$ has a hamiltonian path with both endvertices in $N_H(z)$.

Proof. There exists a hamiltonian cycle in $H + u_1u_2$ which contains the path vzw , where $v, w \in N_H(z)$. Thus there exists a hamiltonian path in $F + u_1u_2$ with endvertices v and w .

■

Construction of the graph $K_4[H_1, H_2, H_3]$

For $i = 1, 2, 3$, let H_i be a cubic MHH graph, with a vertex z_i with neighbours a_i, b_i and c_i , which satisfies the following condition.

Condition (C):

For every vertex $u_i \notin N_{H_i}(z_i)$, the graph $H_i + z_iu_i$ has a hamiltonian cycle containing the edge a_iz_i as well as a hamiltonian cycle not containing a_iz_i .

Graphs satisfying this condition will be presented at the end of the paper.

In the same sense as Grünbaum [8] we use $H_i \setminus z_i$ to denote H_i “opened up” at z_i (see Fig. 2).

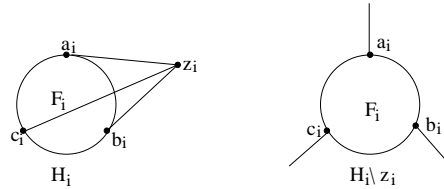


Fig. 2

Let $K_4[H_1, H_2, H_3]$ be an inflated K_4 obtained from $H_i \setminus z_i$; $i = 1, 2, 3$ and a vertex x by joining x to the semi-edge incident with a_i for $i = 1, 2, 3$ and joining the remaining semi-edges as depicted in Fig. 3. Let F_i denote $H_i - z_i$; $i = 1, 2, 3$. We call a_i , b_i and c_i the *exit vertices* of F_i .

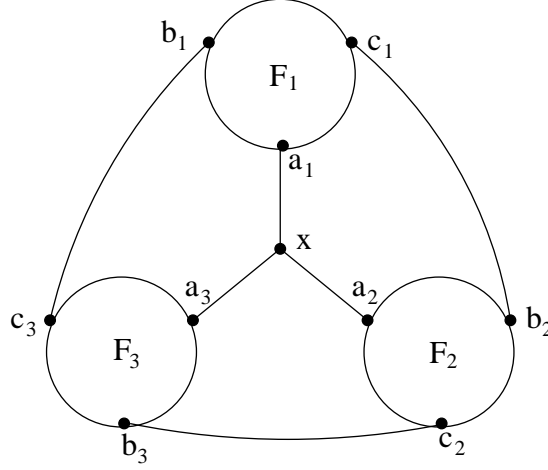


Fig. 3

We introduce the following notation which we use in the theorem below:
 $P_G(v, w)$ denotes a hamiltonian path in a graph G from v to w ;
 $P_G(-, w)$ denotes a hamiltonian path in G ending at w ;
 $P_G(v, -)$ denotes a hamiltonian path in G beginning at v ; and
 P_G denotes a hamiltonian path in G .

Theorem 3.3 *The graph $G = K_4[H_1, H_2, H_3]$ is a cubic MNT graph.*

Proof. It is obvious from the construction that G is cubic.

We now show that G is nontraceable. Suppose P is a hamiltonian path of G . Then at least one of the F_i 's, say F_2 , does not contain an endvertex of P . Thus P passes through F_2 , using two of the exit vertices of F_2 . However, by Lemma 3.1(b) such a path cannot contain all the vertices of F_2 .

We now show that $G + uv$ is traceable for all nonadjacent vertices u and v in G .

Case 1. $u, v \in F_i$; $i \in \{1, 2, 3\}$.

Without loss of generality consider $i = 2$. By Lemma 3.2 there is a hamiltonian path in $F_2 + uv$ with endvertices two of a_2 , b_2 and c_2 .

Subcase (i). Suppose the endvertices are a_2 and c_2 . (A similar proof holds for a_2 and b_2 .) By using Lemma 3.1(a) we obtain the hamiltonian path

$$P_{G+uv} = P_{F_1}(-, a_1)xP_{F_2+uv}(a_2, c_2)P_{F_3}(b_3, -).$$

Subcase (ii). Suppose the endvertices are b_2 and c_2 . By using Lemma 3.1(c) we obtain the hamiltonian path

$$P_{G+uv} = a_1 x P_{F_3-c_3}(a_3, b_3) P_{F_2+uv}(c_2, b_2) P_{G_1-a_1}(c_1, b_1) c_3.$$

Case 2. $u \in \{a_i, b_i, c_i\}$ and $v \in \{a_j, b_j, c_j\}$; $i, j \in \{1, 2, 3\}$; $i \neq j$.

Without loss of generality we choose $i = 2$ and $j = 3$. By using Lemmas 3.1(a) and (c) we find a hamiltonian path P_{G+uv} in $G + uv$. All subcases can be reduced to the following:

Subcase (i). $u = a_2, v = a_3$.

$$P_{G+uv} = a_2 a_3 x P_{G_1-c_1}(a_1, b_1) P_{F_3-a_3}(c_3, b_3) P_{F_2-a_2}(c_2, b_2) c_1.$$

Subcase (ii). $u = a_2, v = b_3$.

$$P_{G+uv} = c_2 b_3 P_{F_2-c_2}(a_2, b_2) P_{G_1-b_1}(c_1, a_1) x P_{F_3-b_3}(a_3, c_3) b_1.$$

Subcase (iii). $u = a_2, v = c_3$.

$$P_{G+uv} = b_1 c_3 P_{F_2-c_2}(a_2, b_2) P_{G_1-b_1}(c_1, a_1) x P_{F_3-c_3}(a_3, b_3) c_2.$$

Subcase (iv). $u = b_2, v = b_3$.

$$P_{G+uv} = c_2 b_3 P_{F_2-c_2}(b_2, a_2) x P_{F_3-b_3}(a_3, c_3) P_{F_1}(b_1, -).$$

Subcase (v). $u = b_2, v = c_3$.

$$P_{G+uv} = c_1 b_2 c_3 P_{G_1-c_1}(b_1, a_1) x P_{F_2-b_2}(a_2, c_2) P_{F_3-c_3}(b_3, a_3).$$

Case 3. $u \in F_i - \{a_i, b_i, c_i\}$ and $v \in F_j$; $i, j \in \{1, 2, 3\}$; $i \neq j$.

Without loss of generality we choose $i = 2$ and $j = 3$. Let F_2^* be the graph obtained from G by contracting $G - V(F_2)$ to a single vertex z_2^* . Then F_2^* is isomorphic to H_2 and hence, by Condition (C), $F_2^* + uz_2^*$ has a hamiltonian cycle containing the path $uz_2^*a_2$. Thus F_2 has a hamiltonian path with endvertices u and a_2 . Using this fact and Lemma 3.1(a) we construct the hamiltonian path

$$P_{G+uv} = P_{F_3}(-, v) P_{F_2}(u, a_2) x P_{F_1}(a_1, -).$$

Case 4. $u = x$ and $v \in F_i$; $i \in \{1, 2, 3\}$.

Without loss of generality we choose $i = 2$.

Subcase (i). $v \in \{b_2, c_2\}$.

Consider $v = b_2$. (The case $v = c_2$ follows similarly.) By using Lemmas 3.1(a) and (c) we obtain the hamiltonian path

$$P_{G+uv} = P_{F_3}(-, b_3) P_{F_2-b_2}(c_2, a_2) x b_2 P_{F_1}(c_1, -).$$

Subcase (ii). $v \in F_2 - \{a_2, b_2, c_2\}$.

According to Condition (C) and an argument similar to that in Case 3, there is a hamiltonian path in F_2 with endvertices v and d , where $d \in \{b_2, c_2\}$. Suppose

$d = b_2$. (A similar proof holds for $d = c_2$.) Using this fact and Lemma 3.1(a) we construct the hamiltonian path

$$P_{G+uv} = P_{F_3}(-, a_3)xP_{F_2}(v, b_2)P_{F_1}(c_1, -).$$

■

The Petersen graph ($n = 10$), the Coxeter graph ($n = 28$) and the Isaacs' snarks J_k ($n = 4k$) for odd $k \geq 5$ are all cubic MHH graphs (see [2], [4]). We determined, by using the Graph Manipulation Package developed by Siqinfu and Sheng Bau*, that a snark of order 22, reported by Chisala [6], is also MHH. The Petersen graph, the snark of order 22 and the Coxeter graph are shown in Fig. 4.

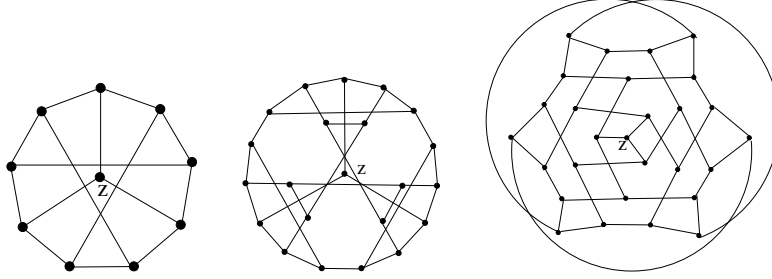


Fig. 4

All the cubic MHH graphs mentioned above satisfy condition **(C)**. In fact, it follows from Theorem 10 in [4] that the Isaac's snarks satisfy a stronger condition, namely that if $v \notin N_H(z)$ then, for *every* $u \in N_H(z)$ there exists a hamiltonian cycle in $H + uv$ containing uz , and this condition holds for all $z \in H$. We determined, again by using the Graph Manipulation Package, that each of the graphs shown in Fig. 4 also satisfy this extended condition for the specified vertex z . The programme allows one to sketch a graph G on the computer screen by placing vertices and adding edges. On request the programme will either draw in a hamiltonian cycle or state that the graph is non-hamiltonian. We tested to see if each of the above mentioned MNH graphs G satisfied the condition **(C)** by considering symmetry and adding appropriate edges and noting the structure of the hamiltonian cycle drawn in.

Thus, by using various combinations of these MHH graphs, we can produce cubic MNT graphs of order

$$n = \begin{cases} 8p & p \geq 5 \\ 8p + 2 & p \geq 6 \\ 8p + 4 & p = 3, p \geq 6^{**} \\ 8p + 6 & p \geq 4. \end{cases}$$

Thus $g_2(n) = \frac{3n}{2}$ for all the values of n stated above.

Remark: Our construction yields MNT graphs of girths 5, 6 and 7. We do not know whether MNT graphs of girth bigger than 7 exist.

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